

INTERSECTION PROPERTIES OF TYPICAL COMPACT SETS

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ABSTRACT. We prove that a typical compact set does not contain any similar copy of a given pattern. We also prove that a typical compact set of $[0, 1]^d$ ($d \geq 2$) intersects any $(d - 1)$ -dimensional plane in at most d points. We study the “hitting probabilities” of compact sets in the sense of Baire category. In the end we study the arithmetic properties of typical compact sets in $[0, 1]$ and the “hitting probabilities” of continuous functions.

1. INTRODUCTION

A subset of a metric space X is of first category if it is a countable union of *nowhere dense* sets (i.e. whose closure in X has empty interior); otherwise it is called of second category. We say that a *typical* element $x \in X$ has property P , if the complement of

$$\{x \in X : x \text{ satisfies } P\}$$

is of first category. For the basic properties and various applications of Baire Category, we refer to [13, 17]. Let $\mathcal{K} = \mathcal{K}([0, 1]^d)$ be all the compact subsets of unite cube $[0, 1]^d$. We endow \mathcal{K} with *Hausdorff metric*. Recall that the Hausdorff distance of two compact sets E and F of \mathcal{K} is defined by

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F^\varepsilon \text{ and } F \subset E^\varepsilon\},$$

where $E^\varepsilon = \{x \in \mathbb{R}^d : \text{dist}(x, E) < \varepsilon\}$.

Davies, Mastrand and Taylor [5] constructed a compact set $A \subset [0, 1]$ with Hausdorff dimension zero containing a similar copy of any finite set. Chen and Rossi [4] showed that a typical compact set is locally rich which means that we can “see” all the compact sets when we zooming in at any point of this compact set. Feng and Wu [7] proved that a typical compact set has Hausdorff dimension zero. It is natural to ask

Date: December 16, 2015.

2000 Mathematics Subject Classification. Primary 28A80, Secondary 37F40.

Key words and phrases. Baire category, Hausdorff metric, intersection.

The author acknowledged the support of the Academy of Finland, the Centre of Excellence in Analysis and Dynamics Research.

that does a typical compact set containing a similar copy of any finite set. We have the following negative answer.

Theorem 1.1. *A typical compact set does not contain a similar copy of a given set P with three distinct points.*

Note that the Lebesgue density theorem implies that any set of \mathbb{R}^d with positive Lebesgue measure contains a similar copy of any finite set. However, Keleti [8, 9] constructed an 1-dimensional compact set that does not contain the non-trivial 3-term arithmetic progressions. Recently, Shmerkin [15] constructed an 1-dimensional Salem set without 3-term arithmetic progressions also. For more backgrounds and further results we refer to [1, 3, 10, 16]. For the basic properties of Hausdorff dimension we refer to [6, 12].

It is not hard to see that if the complement of $A \subset \mathbb{R}^d$ is of first category, then A contains a similar copy of any countable set. This follows by the fact that for any countable set $\{t_i \in \mathbb{R}^d : i \in \mathbb{N}\}$, the intersection $\bigcap_{i \in \mathbb{N}} (A + t_i)$ is not empty. Note that A is not a compact set. However, there exists a second category set E in the plane such that any line intersects E in at most two points, see [13, Theorem 15.5]. For a typical compact set of \mathcal{K} we have the following result.

Theorem 1.2. *A typical compact set of $\mathcal{K}([0, 1]^d)$ ($d \geq 2$) intersects any $(d - 1)$ -dimensional plane in at most d points.*

Let $A \subset [0, 1]^d$ and $\mathcal{K}_A = \{E \in \mathcal{K} : E \cap A \neq \emptyset\}$. It is reasonable to think that if A is a “small” set in $[0, 1]^d$ then \mathcal{K}_A will be a “small” set in \mathcal{K} also.

Theorem 1.3. *A set $A \subset [0, 1]^d$ is nowhere dense in $[0, 1]^d$ if and only if \mathcal{K}_A is nowhere dense in \mathcal{K} .*

If $A \subset [0, 1]^d$ is of first category in $[0, 1]^d$, then $A = \bigcup_{i \in \mathbb{N}} A_i$ where each A_i is nowhere dense in $[0, 1]^d$. Observe that

$$\mathcal{K}_A = \bigcup_{i \in \mathbb{N}} \mathcal{K}_{A_i}.$$

Theorem 1.3 claims that \mathcal{K}_{A_i} is nowhere dense in \mathcal{K} for each $i \in \mathbb{N}$, and hence \mathcal{K}_A is of first category in \mathcal{K} . It follows that a typical compact set of \mathcal{K} does not intersects A . Šalát [14] proved that the set of *normal numbers* is of first category. It is also known that the complementary set of *Liouville numbers* is of first category, see [13, Chapter 2]. Thus we obtain that a typical compact set of $\mathcal{K}([0, 1])$ is a subset of non-normal Liouville numbers. We collect these facts as the following corollary.

Corollary 1.4. (a) *If $A \subset [0, 1]^d$ is of first category in $[0, 1]^d$ then \mathcal{K}_A is of first category in \mathcal{K} .*

(b) *A typical compact set of $\mathcal{K}([0, 1])$ is a subset of non-normal Liouville numbers.*

We do not know that whether \mathcal{K}_A is of first category implies that A is of first category.

In the following, we study the size of sets formulated under finite steps arithmetic operations of a typical set A of $\mathcal{K}([0, 1])$. Let $S^m(A)$ be the m -th sum set of A , and $\tilde{P}(A)$ be a set formed under the rule of the polynomial P . We show these definitions in Section 4. Under these notations we have the following result.

Theorem 1.5. *For a typical compact set A of $\mathcal{K}([0, 1])$, we have that $\dim_H S^m(A) = 0$ for any $m \in \mathbb{N}$ and $\dim_H \tilde{P}(A) = 0$ for any polynomial P .*

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in section 2. Theorem 1.3 is proved in section 3. Theorem 1.5 is proved in Section 4. In the end we study the “hitting probabilities” of continuous function.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Let $A, B \subset \mathbb{R}^d$. If there exist $\lambda > 0$ and an isometric map φ on \mathbb{R}^d such that $\varphi(\lambda A) = B$, then we say that A is similar to B and denote this by $A \sim B$. If there is a subset $A' \subset A$ such that $A' \sim B$, then we say that A contains a similar copy of B . For each $n \in \mathbb{N}$, let $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ where \mathcal{D}_n is the family of 2^n -adic closed subcubes of $[0, 1]^d$, i.e.

$$\mathcal{D}_n = \left\{ \prod_{k=1}^d [i_k 2^{-n}, (i_k + 1) 2^{-n}] : 0 \leq i_k \leq 2^n - 1 \right\}.$$

For a set $A \subset \mathbb{R}^d$, denote by ∂A the boundary of A , denote by $|A|$ the diameter of A . For two points $x, y \in \mathbb{R}^d$, denote by $|x - y|$ the Euclidean metric. Denote by \mathcal{L}^d the d -dimensional Lebesgue measure. Let $\{x_1, x_2, x_3\} \subset \mathbb{R}^d$ be three distinct points. Define

$$R(x_1, x_2, x_3) = \left\{ \frac{|x_i - x_k|}{|x_j - x_k|} : i \neq j, j \neq k, i \neq k \right\}.$$

Lemma 2.1. *Let $P = \{p_1, p_2, p_3\}$ be three distinct points of \mathbb{R}^d and $a, b \in \mathbb{R}^d, a \neq b$. Then $\mathcal{L}^d(P') = 0$, where*

$$P' = \{x \in \mathbb{R}^d : P \sim \{a, b, x\}\}.$$

Proof. If $P \sim \{a, b, x\}$, then $R(p_1, p_2, p_3) = R(a, b, x)$. It follows that there are at most $N = N(d)$ balls $\{B_i\}_{i=1}^N$ such that $P' \subset \cup_{i=1}^N \partial B_i$, and hence $\mathcal{L}^d(P') = 0$. \square

Lemma 2.2. *Let $P = \{p_1, p_2, p_3\}$ be three distinct points of $[0, 1]^d$. Then for any $n \in \mathbb{N}$, there exists $\Gamma_n = \{x_Q : Q \in \mathcal{D}_n\}$ with $x_Q \in Q$ such that any three distinct points of Γ_n is not similar to P . Moreover there exists $\varepsilon = \varepsilon_n$ such that the following two conditions hold.*

(C₁) $B(x_Q, \varepsilon) \subset Q$ for each $Q \in \mathcal{D}_n$.

(C₂) For any $\{a_1, a_2, a_3\} \subset \bigcup_{Q \in \mathcal{D}_n} B(x_Q, \varepsilon)$ which is similar to P , there exists $Q \in \mathcal{D}_n$ such that $\{a_1, a_2, a_3\} \subset B(x_Q, \varepsilon)$.

Proof. Let $\mathcal{D}_n = \{Q_i : 1 \leq i \leq 2^{nd}\}$. Assume that we have chosen m points $K_m = \{x_i : 1 \leq i \leq m\}$ with $x_i \in Q_i, 1 \leq i \leq m$ such that any three distinct points of K_m is not similar to P . For any two points x_i, x_j of K_m , by Lemma 2.1 we obtain that the set

$$\{x \in \mathbb{R}^d : P \sim \{x_i, x_j, x\}\}$$

has Lebesgue measure zero. Note that there are at most $m(m-1)/2$ pairs of (i, j) . It follows that there exists an interior point x_{m+1} of Q_{m+1} such that any three points of $\{x_i : 1 \leq i \leq m+1\}$ is not similar to P . We use the same way to find points x_{m+2}, \dots . In the end, we obtain a point $x_{2^{dn}}$ from $Q_{2^{dn}}$. Let Γ_n be the collection of chosen points.

Since each x_i is an interior point of $Q_i, 1 \leq i \leq 2^{nd}$, there is $\varepsilon' > 0$ such that the condition C₁ holds. Observe that for any three distinct points $\{x_{i_1}, x_{i_2}, x_{i_3}\} \subset \Gamma_n$, there is a positive constant $\varepsilon_{i_1, i_2, i_3}$ such that any three points $\{a_1, a_2, a_3\}$ with $a_k \in B(x_{i_k}, \varepsilon_{i_1, i_2, i_3}), k = 1, 2, 3$ is not similar to P . Let ε'' be the minimal value over all the possible $\varepsilon_{i_1, i_2, i_3}$, and $\varepsilon = \min\{\varepsilon', \varepsilon''\}$. Thus we complete the proof. \square

Definition 2.3. Recall that the points $\{x_1, x_2, \dots, x_m\}$ are called affinely independent if the vectors $x_2 - x_1, \dots, x_m - x_1$ are linearly independent. Let $A \subset \mathbb{R}^d, d \geq 2$. We say A is affinely independent if any $k(3 \leq k \leq d+1)$ distinct points of A is affinely independent.

Let $A \subset \mathbb{R}^d, d \geq 2$ be a set that intersects any $(d-1)$ -dimensional plane in at most d points. Observe that this is equivalent to say that any $(d+1)$ points $\{x_1, \dots, x_{d+1}\} \subset A$ is affinely independent.

Lemma 2.4. *For each $\mathcal{D}_n, n \in \mathbb{N}$, there exists $\Gamma_n = \{x_Q : Q \in \mathcal{D}_n\}$ with $x_Q \in Q$ such that Γ_n is affinely independent. Moreover there exists $\varepsilon = \varepsilon_n$ such that the following two conditions hold.*

(C₁) $B(x_Q, \varepsilon) \subset Q$ for each $Q \in \mathcal{D}_n$.

(C₂) For any $\{a_1, \dots, a_{d+1}\} \subset \bigcup_{Q \in \mathcal{D}_n} B(x_Q, \varepsilon)$ which is not affinely independent, there exists $Q \in \mathcal{D}_n$ and $\{a_i, a_j\} \subset \Gamma_n$ such that $\{a_i, a_j\} \subset B(x_Q, \varepsilon)$.

Proof. Let $\mathcal{D}_n = \{Q_i : 1 \leq i \leq 2^{nd}\}$. Assume that we have chosen m points $K_m = \{x_i : 1 \leq i \leq m\}$ with $x_i \in Q_i, 1 \leq i \leq m$ such that K_m is affinely independent. Observe that for any d points $x_{i_k}, 1 \leq k \leq d$ of K_m , the set

$$\{x \in \mathbb{R}^d : \{x\} \cup \{x_{i_k} : 1 \leq k \leq d\} \text{ is not affinely independent} \}$$

has Lebesgue measure zero. Note that there are at most finite elements of (i_1, i_2, \dots, i_d) . It follows that there exists an interior point x_{m+1} of Q_{m+1} such that $\{x_i : 1 \leq i \leq m+1\}$ is affinely independent. We use the same way to find points x_{m+2}, \dots . In the end, we obtain a point $x_{2^{dn}}$ from $Q_{2^{dn}}$. Let Γ_n be the collection of chosen points.

Since each x_i is an interior point of $Q_i, 1 \leq i \leq 2^{nd}$, there is $\varepsilon' > 0$ such that the condition C_1 holds. Observe that for any $d+1$ distinct points $\{x_{i_1}, \dots, x_{i_{d+1}}\} \subset \Gamma_n$, there is a positive constant $\varepsilon_{i_1, \dots, x_{i_{d+1}}}$ such that any $(d+1)$ points $\{a_1, \dots, a_{d+1}\}$ with

$$a_k \in B(x_{i_k}, \varepsilon_{i_1, \dots, x_{i_{d+1}}}), 1 \leq k \leq d+1$$

is not affinely independent. Let ε'' be the minimal value over all the possible $\varepsilon_{i_1, \dots, x_{i_{d+1}}}$ and $\varepsilon = \min\{\varepsilon', \varepsilon''\}$. \square

In fact we can also choose the sets Γ_n of Lemma 2.2 and Lemma 2.4 in a probability way. For each $Q \in \mathcal{D}_n$, we randomly choose a point $x_Q \in Q$ under the law of uniform distribution. The choices are independent for different cubes of \mathcal{D}_n . Denote by Γ_n^ω the random chosen points. It is not hard to show that with probability one Γ_n^ω has the same properties as Γ_n . We show the outline for this argument.

Proposition 2.5. *With probability one Γ_n^ω has the same properties as Γ_n in Lemma 2.2 and Lemma 2.4.*

Proof. Let $\mathcal{D}_n = \{Q_1, Q_2, \dots, Q_{2^{nd}}\}$. Lemma 2.1 implies that conditional on $x_1 \in Q_1, x_2 \in Q_2$, the probability of the event $P \sim \{x_1, x_2, x_3\}$ is zero. Therefore we have that $\mathbb{P}(P \sim \{x_1, x_2, x_3\}) = 0$. It follows that

$$\begin{aligned} & \mathbb{P}(\text{exists } \{x_i, x_j, x_k\} \subset \Gamma_n^\omega \text{ such that } \{x_i, x_j, x_k\} \sim P) \\ & \leq \sum_{i,j,k} \mathbb{P}(\{x_i, x_j, x_k\} \sim P) = 0. \end{aligned}$$

Thus we obtain that with probability one any three points of Γ_n^ω is not similar to P .

Observe that

$$\mathbb{P}(\{x_1, x_2, x_3\} \text{ is affinely independent}) = 1.$$

Let A_k be the event

$$\{x_1, x_2, \dots, x_k\} \text{ is affinely independent, } 3 \leq k \leq d+1.$$

Then it is not hard to see that $\mathbb{P}(A_{k+1}|A_k) = 1$, and $\mathbb{P}(A_{k+1}|A_k^c) = 0$. Thus we have

$$\begin{aligned} \mathbb{P}(A_{d+1}) &= \mathbb{P}(A_{d+1}|A_d)\mathbb{P}(A_d) + \mathbb{P}(A_{d+1}|A_d^c)\mathbb{P}(A_d^c) \\ &= \mathbb{P}(A_d) = \dots = \mathbb{P}(A_3) = 1. \end{aligned}$$

It follows that $\mathbb{P}(A_{d+1}^c) = 0$, and thus

$$\begin{aligned} &\mathbb{P}(\Gamma_n^\omega \text{ is not affinely independent}) \\ &\leq \sum_{i_1, \dots, i_{d+1}} \mathbb{P}(\{x_{i_1}, \dots, x_{i_{d+1}}\} \text{ is not affinely independent}) = 0. \end{aligned}$$

Since the boundary of cube has Lebesgue measure zero, we obtain that with probability one x_i is an interior point of Q_i , $1 \leq i \leq 2^{nd}$. \square

Proof of Theorem 1.1. Let $P = \{p_1, p_2, p_3\} \subset [0, 1]^d$. For each $n \in \mathbb{N}$, let Γ_n be the set in Lemma 2.2 and P_n be the power set of Γ_n . Recall that the power set of a set X is the collection of all the subset of X . Let

$$\mathcal{G} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n),$$

where $U_{d_H}(\gamma, \varepsilon_n)$ is an open set of (\mathcal{K}, d_H) with center γ and radius ε_n . Note that $\{\gamma : \gamma \in P_n, n \in \mathbb{N}\}$ is a countable dense subset in \mathcal{K} . Thus

$$\bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n)$$

is a dense open set in \mathcal{K} . It follows that the complementary set of \mathcal{G} is of first category. In the following we intend to show that any element of \mathcal{G} does not contain a similar copy of $\{p_1, p_2, p_3\}$.

Let $E \in \mathcal{G}$, then there exist $n_k \nearrow \infty$ and $\gamma_{n_k} \in P_{n_k}$ such that $E \in \bigcap_{k=1}^{\infty} U_{d_H}(\gamma_{n_k}, \varepsilon_{n_k})$. Suppose that there is $\{x_1, x_2, x_3\} \subset E$ which is similar to F . By the condition C_2 of Lemma 2.2, there is $Q \in \mathcal{D}_{n_k}$ such that

$$\{x_1, x_2, x_3\} \subset B(x_Q, \varepsilon_{n_k}),$$

and hence

$$|\{x_1, x_2, x_3\}| \leq 2\varepsilon_{n_k} \leq \sqrt{d}2^{-n_k} \rightarrow 0.$$

This is a contradiction. Thus we complete the proof. \square

Proof of Theorem 1.2. For each $n \in \mathbb{N}$, let Γ_n be the set in Lemma 2.4 and P_n be the power set of Γ_n . Let

$$\mathcal{G} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n).$$

Then the complementary set of \mathcal{G} is of first category.

Let $E \in \mathcal{G}$, then there exist $n_k \nearrow \infty$ and $\gamma_{n_k} \in P_{n_k}$ such that $E \in \bigcap_{k=1}^{\infty} U_{d_H}(\gamma_{n_k}, \varepsilon_{n_k})$. Suppose that E is not affinely independent. Thus there exists $\{a_1, \dots, a_{d+1}\} \subset E$ such that $\{a_1, \dots, a_{d+1}\}$ is not affinely independent. For each n_k , there exists $\gamma \in \Gamma_n$ such that

$$\{a_1, \dots, a_{d+1}\} \subset \bigcup_{x \in \gamma} B(x, \varepsilon_n).$$

By the condition C_2 of Lemma 2.4, we obtain that there exists two distinct points a_i, a_j with $|a_i, a_j| \leq 2\varepsilon_n$. Note that we may choose ε_n such that $\varepsilon_n \searrow 0$. It follows that there exist two points of $\{a_1, \dots, a_{d+1}\}$ with distance zero which is a contradiction. \square

Remark 2.6. Let $E \subset \mathbb{R}^d, d \geq 2$. We say E contains the angle θ if there are three points $\{x, y, z\} \subset E$ such that the angle between the vectors $y - x$ and $z - x$ is θ , and write $\angle \theta \in E$. For some results on this topic and further references we refer to [3, 16].

Let $\theta \in [0, \pi)$ and $a, b \in \mathbb{R}^d, d \geq 2$. Then by some elementary geometric arguments, we have

$$\mathcal{L}^d(\{x \in \mathbb{R}^d : \angle \theta \in \{a, b, x\}\}) = 0.$$

It follows that for each $n \in \mathbb{N}$, there is $\Gamma_n = \{x_Q : Q \in \mathcal{D}_n\}$ such that Γ_n does not contain the angle θ . Applying the similar argument in the proofs of Theorems 1.1 and 1.2, we obtain that a typical compact set of $\mathcal{K}([0, 1]^d), d \geq 2$ does not contain the angle θ . We omit the details here.

3. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. Suppose A is a nowhere dense subset of $[0, 1]^d$. Let $E \in \mathcal{K}, \varepsilon = 2^{-n}\sqrt{d}$. Assume first that $E \cap A \neq \emptyset$. We define

$$\mathcal{E}_n = \{Q \in \mathcal{D}_n : Q \cap E \neq \emptyset\} = \mathcal{E}'_n \cup \mathcal{E}''_n$$

where

$$\mathcal{E}'_n = \{Q \in \mathcal{E}_n : Q \cap A = \emptyset\}, \mathcal{E}''_n = \mathcal{E}_n \setminus \mathcal{E}'_n.$$

For every $Q \in \mathcal{E}'_n$, let c_Q be the center point of Q . For every $Q \in \mathcal{E}''_n$, since A is nowhere dense, there exists $x_Q \in Q, r_Q > 0$ such that

$$U(x_Q, r_Q) \subset Q \text{ and } U(x_Q, r_Q) \cap A \neq \emptyset.$$

Let F be the collection of points c_Q for $Q \in \mathcal{E}'_n$ and x_Q for $Q \in \mathcal{E}''_n$. Then $F \in U_{d_H}(E, 2^{-n}\sqrt{d})$. Let

$$\varepsilon' = \min\{r_Q : Q \in \mathcal{E}''_n\}.$$

Then $U_{d_H}(F, \varepsilon') \cap \mathcal{K}_A = \emptyset$.

For the case $E \cap A = \emptyset$ we have that $\mathcal{E}'' = \emptyset$. Let F be the collection of points c_Q for $Q \in \mathcal{E}'_n$. Then

$$F \in U_{d_H}(E, 2^{-n}\sqrt{d}) \text{ and } U_{d_H}(F, 2^{-n-1}) \cap \mathcal{K}_A = \emptyset.$$

By the arbitrary choice of $E \in \mathcal{K}$ and $\varepsilon = 2^{-n}\sqrt{d}$, we obtain that \mathcal{K}_A is nowhere dense in \mathcal{K} .

Now we assume that there is an open ball $U \subset \overline{A}$ where \overline{A} is the closure of A . Let $\mathcal{K}(U)$ be all the compact subsets of U , then $\mathcal{K}(U)$ is an open set in \mathcal{K} . Observe that $\mathcal{K}(U) \subset \overline{\mathcal{K}_A}$. Thus we obtain that if \mathcal{K}_A is nowhere dense in \mathcal{K} then A is nowhere dense in $[0, 1]^d$. \square

4. PROOF OF THEOREM 1.5

For $A, B \subset \mathbb{R}$ we define their sum set

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let $\lambda \in \mathbb{R}$ and $\lambda A = \{\lambda \times a : a \in A\}$. For $m \in \mathbb{N}$, define

$$S^m(A) := \left\{ \sum_{i=1}^m x_i : x_i \in A, 1 \leq i \leq m \right\},$$

$$A^m = \{(x_1, \dots, x_m) : x_i \in A, 1 \leq i \leq m\},$$

and

$$T^m(A) := \{x_1 \times \dots \times x_m : x_i \in A, 1 \leq i \leq m\}.$$

Let $P(x) = \sum_{k=0}^n a_k x^k$, $a_k \in \mathbb{R}$ be a polynomial. For a set $A \subset \mathbb{R}$, let

$$\tilde{P}(A) := \sum_{k=0}^n a_k T^k(A).$$

Note that $\tilde{P}(A)$ is the sum set of $\{a_k T^k(A)\}_{k=0}^n$. The Hausdorff dimension of E is defined as

$$\dim_H E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\},$$

where $\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E)$, and

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subset \bigcup_{i \in \mathbb{N}} U_i, |U_i| \leq \delta, i \in \mathbb{N} \right\}.$$

For each $n \in \mathbb{N}$, let $\varepsilon_n = 2^{-n^2}$,

$$\mathcal{D}'_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\},$$

and P_n be the power set of \mathcal{D}'_n . Define

$$\mathcal{G} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \bigcup_{\gamma \in P_n} U_{d_H}(\gamma, \varepsilon_n).$$

Applying the same argument as in the proof of Theorem 1.1, we have that the complementary of \mathcal{G} is of first category in \mathcal{K} .

Lemma 4.1. *Let $A \in \mathcal{G}$ then $\dim_H A^m = 0$ for any $m \in \mathbb{N}$.*

Proof. For any $k \in \mathbb{N}$ there exist $n \geq k$ and

$$\gamma = \{x_1, \dots, x_N\} \in P_n$$

such that $A \in U_{d_H}(\gamma, \varepsilon_n)$. It follows that

$$A \subset \bigcup_{i=1}^N B_i$$

where $B_i := B(x_i, \varepsilon_n)$, $1 \leq i \leq N$. Let $m \in \mathbb{N}$, then

$$A^m \subset \bigcup_{i_1 \dots i_m \in \mathcal{I}^m} B_{i_1} \times \dots \times B_{i_m}$$

where $\mathcal{I} = \{1, 2, \dots, N\}$. Note that

$$|B_{i_1} \times \dots \times B_{i_m}| \leq \sqrt{m} \varepsilon_n \text{ for any } i_1 \dots i_m \in \mathcal{I}^m.$$

Since $N \leq 2^n + 1$ and $\varepsilon_n = 2^{-n^2}$, for any $s > 0$ we have

$$\mathcal{H}_{\varepsilon_n \sqrt{m}}^s(A^m) \leq N^m (\varepsilon_n \sqrt{m})^s \leq 2^{n+1} 2^{-n^2 s} \sqrt{m}^s.$$

It follows that $\mathcal{H}^s(A^m) = 0$. By the arbitrary choice of $s > 0$ we have that $\dim_H A^m = 0$. Thus we complete the proof. \square

Proof of Theorem 1.5. It is clear that $S^m(A) = \sqrt{m} \pi_e(A^m)$ where $\pi_e(A^m)$ is the orthogonal projection of A^m on to the line with direction $e = \sqrt{m}^{-1}(1, 1, \dots, 1)$. Thus $\dim_H S^m(A) \leq \dim_H A^m$. Therefore by Lemma 4.1 we obtain $\dim_H S^m(A) = 0$.

Suppose that

$$P(x) = \sum_{k=0}^n a_k x^k, a_k \in \mathbb{R}, a_n \neq 0.$$

Does not lose general we may assume $a_0 = 0$. Note that

$$\tilde{P}(A) = \left\{ \sum_{k=1}^n a_k x_{k,1} \cdots x_{k,k} : 1 \leq i \leq k, x_{k,i} \in A \right\}.$$

Define a new function

$$\varphi : [0, 1]^{\frac{(n+1)n}{2}} \longrightarrow \mathbb{R}$$

by

$$\varphi(x_{1,1}, x_{2,1}, x_{2,2}, \cdots, x_{nn}) = \sum_{k=1}^n a_k x_{k,1} \cdots x_{k,k}.$$

By the mean value theorem we have that φ is a Lipschitz map on $[0, 1]^{\frac{(n+1)n}{2}}$. Observe that

$$\tilde{P}(A) = \varphi(A^{\frac{(n+1)n}{2}}).$$

Thus by Lemma 4.1 and the fact that Lipschitz map will not increase the Hausdorff dimension, we obtain that $\dim_H \tilde{P}(A) = 0$. \square

Remark 4.2. Let $A \subset \mathbb{R}^d$. Then we can also consider the sets $S^m(A)$, A^m . By applying the same arguments in Lemma 4.1 and in the proof of Theorem 1.5, we have that for a typical compact set $A \in \mathcal{K}([0, 1]^d)$,

$$\dim_H A^m = 0 \text{ and } \dim_H S^m(A) = 0 \text{ for any } m \in \mathbb{N}.$$

We omit the details here.

Denote

$$e^A := \sum_{n=0}^{\infty} \frac{T^n(A)}{n!}.$$

We consider e^A as the limit point of S_m in the space $(\mathcal{K}(\mathbb{R}), d_H)$ where

$$S_m := \sum_{n=0}^m \frac{T^n(A)}{n!}$$

is a sum set of $\{\frac{T^n(A)}{n!}\}_{n=0}^m$. Note that $\{S_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $(\mathcal{K}(\mathbb{R}), d_H)$. Thus the set e^A is well defined.

Question 4.3. *Is it true that a typical $A \in \mathcal{K}$ has $\dim_H e^A = 0$?*

5. TYPICAL CONTINUOUS FUNCTIONS

Let $\mathcal{C} = \mathcal{C}([0, 1])$ be all the continuous functions on $[0, 1]$. The distance of continuous functions $f, g \in \mathcal{C}$ is defined by

$$d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\}.$$

Let $A \subset [0, 1] \times \mathbb{R}$. Define

$$\mathcal{C}_A = \{f \in \mathcal{C} : G(f) \cap A \neq \emptyset\}$$

where $G(f)$ is the graph of function f . Let $x \in [0, 1]$ and $V_x = \{x\} \times \mathbb{R}$. In the following we only consider the case $A \subset V_x$. For this special case, we have the following similar result to Theorem 1.3.

Proposition 5.1. *A subset $A \subset V_x, x \in [0, 1]$ is nowhere dense in V_x if and only if \mathcal{C}_A is nowhere dense in \mathcal{C} .*

Proof. Suppose A is nowhere dense in V_x . Let $U_{\mathcal{C}}(f, \varepsilon)$ be an open ball in \mathcal{C} with center f and radius ε . Then by the nowhere dense of A there exist $g \in \mathcal{C}, \varepsilon' > 0$ such that

$$U(g(x), \varepsilon') \cap A = \emptyset, \text{ and } U(g(x), \varepsilon') \subset U(f(x), \varepsilon).$$

Here $U(f(x), \varepsilon)$ is an open ball in V_x with center $f(x)$ and radius ε . Note that $U_{\mathcal{C}}(g, \varepsilon') \cap \mathcal{C}_A = \emptyset$. By the arbitrary choice of $f \in \mathcal{C}$ and ε we obtain that \mathcal{C}_A is nowhere dense.

By applying the same argument as in the proof of Theorem 1.3, we obtain that if \mathcal{C}_A is nowhere dense then A is nowhere dense. \square

Applying the same argument as in the introduction, we obtain that if $A \subset V_x, x \in [0, 1]$ is of first category in V_x then \mathcal{C}_A is of first category in \mathcal{C} . Again we do not know that if the converse claim is also true. Let $z \in [0, 1] \times \mathbb{R}$ then Proposition 5.1 claims that \mathcal{C}_z is nowhere dense in \mathcal{C} . Since the rational points in plane is countable, we obtain that the graph of a typical continuous function of \mathcal{C} does not contain any rational points in plane.

Maga [11] proved that for any distinct points $\{x, y, z\} \subset \mathbb{R}^2$, there exists a compact set $E \subset \mathbb{R}^2$ with $\dim_H E = 2$ and E does not contain a similar copy of $\{x, y, z\}$. Motived by this result and Theorem 1.1 we ask the following question.

Question 5.2. *Does the graph of a continuous function with Hausdorff dimension larger than one contains three points which are the vertices of an equilateral triangle?*

Acknowledgements. I thank Ville Suomala for helpful discussions, and Meng Wu and Wen Wu for valuable comments.

REFERENCES

- [1] M. Bennett, A. Iosevich, K. Taylor. Finite Chains inside Thin Subsets of \mathbb{R}^d . Available at <http://arxiv.org/abs/1409.2581>.
- [2] Y. Bugeaud. Distribution modulo one and Diophantine approximation. Cambridge Tracts in Mathematics 193, Cambridge, 2012.
- [3] V. Chan, I. Laba and M. Pramanik. Point configurations in sparse sets. Journal d'Analyse Mathématique, To appear, 2014. available at <http://arxiv.org/abs/1307.1174>
- [4] C. Chen and E. Rossi. Locally rich compact sets, Illinois J. Math. Volume 58, Number 3 (2014), 779-806.
- [5] R. O. Davies, J. M. Marstrand and S. J. Taylor. On the intersections of transforms of linear sets, Colloquium Mathematicum, 7 (1960), 237-243.
- [6] K. J. Falconer. Fractal Geometry: Mathematical Foundations and Applications, John Wiley, 2nd Ed., 2003.
- [7] D. Feng and J. Wu. Category and dimension of compact subsets of \mathbb{R}^d , Chinese Sci. Bull. 42 (1997), no. 20, 1680-1683. MR 1613815 (99d:28009)
- [8] T. Keleti. A 1-dimensional subset of the reals that intersects each of its translates in at most a single point. Real Anal. Exchange, 24(2):843-844, 1998/99.
- [9] T. Keleti. Construction of one-dimensional subsets of the reals not containing similar copies of given patterns. Anal. PDE, 1(1):29-33, 2008.
- [10] I. Laba and M. Pramanik. Arithmetic progressions in sets of fractional dimension. Geom. Funct. Anal., 19(2):429-456, 2009.
- [11] P. Maga. Full dimensional sets without given patterns. Real Anal. Exchange, 36, 79-90, 2010.
- [12] P. Mattila. Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, 1995.
- [13] J. Oxtoby. Measure and Category, New York: Springer-Verlag, 1980.
- [14] T. Šalát. A remark on normal numbers, Rev. Roumaine Math. Pures Appl. 11 (1966), 53-56.
- [15] P. Shmerkin. Salem sets with no arithmetic progressions. Available at <http://arxiv.org/abs/1510.07596>.
- [16] P. Shmerkin and V. Suomala. Arithmetic structure in random fractal sets, work in progress.
- [17] E. M. Stein, R. Shakarchi. Functional Analysis: An Introduction to Further Topics in Analysis. Princeton University Press, 2011.

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